Fifty Integrals

(for those who play integration as a sport)

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These problems are closed, conditional, open, formal, or elementary. There's no order of difficulty.

Warning. Some are exceptionally hard.

Evaluate, regularize, or characterize as appropriate. Principal branches are used unless stated otherwise. The symbols $[u^r v^s]$, $\int^{(q)}$, $\{\cdot\}_p$, F. p., and det$'$ denote coefficient extraction at $u = v = 0$, the Jackson integral, the p-adic fractional part, Hadamard finite parts at the displayed singularities, and omission of zero modes. Also $\operatorname{Arg} \in (-\pi, \pi]$; limits with $\varepsilon \downarrow 0$ are Abel limits.

Solutions, corrections, and serious questions: *suraj@globaltindia.com.* Please have enough courtesy to avoid things that are extra frivilous. Here you go:

1. $$I_n(x) = \int \tan x \left(a_1 + \sqrt{a_2 + \sqrt{\cdots + \sqrt{a_n + b\cos x}}}\right)^{\rho} \mathrm{d}x, \qquad n \in \mathbb{N}$$

2. $$I = \int x^m (a + bx^n)^p \mathrm{e}^{qx^r} \log^k x \; \mathrm{d}x, \qquad (m, n, p, r) \in \mathbb{Q}^4, \quad k \in \mathbb{N}_0, \quad a, b, q \in \mathbb{C}$$

3. $$I = \int_0^{\infty} x^s \prod_{j=1}^{p} J_{\nu_j}(a_j x) \prod_{k=1}^{q} K_{\mu_k}(b_k x) \; \mathrm{d}x, \qquad a_j, b_k > 0$$

4. $$I_q(\lambda) = \int_{[0,1]^n}^{(q)} P_\lambda(\mathbf{x}; q, t) \prod_{i=1}^{n} x_i^{\alpha-1} \frac{(qx_i; q)_\infty}{(q^\beta x_i; q)_\infty} \prod_{i<j} x_i^{2\tau} \frac{(q^{1-\tau} x_j/x_i; q)_\infty}{(q^\tau x_j/x_i; q)_\infty} \; \mathrm{d}_q\mathbf{x}, \quad 0 < q < 1, \quad t = q^\tau, \quad \tau > 0$$

5. $$I_{pqr}(\omega) = \mathrm{F.\,p.}\int_0^1 \frac{\log^p x \, \log^q(1-x) \, \log^r(1-\omega x)}{x(1-x)} \; \mathrm{d}x, \qquad (p, q, r) \in \mathbb{N}_0^3, \quad \omega \in \mathbb{C} \setminus [1, \infty)$$

6. $$I_{mpqr} = \int_0^1 \frac{\log^p x \, \log^q(1-x)}{(1 + x + \cdots + x^{m-1})^r} \; \mathrm{d}x, \qquad m \geq 2, \quad (p, q, r) \in \mathbb{N}_0^2 \times \mathbb{N}$$

7. $$I_m = \int_0^1 x^{-x}(1-x)^{-(1-x)} \log^m \frac{x}{1-x} \; \mathrm{d}x, \qquad m \in \mathbb{N}_0$$

8. $$I(\alpha, \beta, \lambda) = \int_0^1 x^{-x^\alpha}(1-x)^{-\lambda(1-x)^\beta} \; \mathrm{d}x, \qquad (\alpha, \beta, \lambda) \in \mathbb{R}^3_{>0}$$

9. $$I_{nm} = \lim_{\varepsilon \downarrow 0} \int_0^{\infty} \mathrm{e}^{-\varepsilon x} \left(\frac{\sin x}{x}\right)^n \log^m x \; \mathrm{d}x, \qquad (n, m) \in \mathbb{N} \times \mathbb{N}_0$$

10. $$I_{2,2}(a, b, c, d) = [u^2 v^2] \, \frac{1}{2\pi \mathrm{i}} \int_{\sigma - \mathrm{i}\infty}^{\sigma + \mathrm{i}\infty} \Gamma(a + u + s)\Gamma(b - u + s)\Gamma(c + v - s)\Gamma(d - v - s) \; \mathrm{d}s,$$
$$a, b, c, d > 0, \qquad -\min(a, b) < \sigma < \min(c, d)$$

11. $$I_m(s; \alpha, \beta, \gamma) = \int_0^{\infty} \frac{x^{s-1} \log^m(1 + x^\alpha)}{(1 + x^\beta)^\gamma} \; \mathrm{d}x, \qquad m \in \mathbb{N}_0, \quad \alpha, \beta, \gamma > 0, \quad -m\alpha < \Re s < \beta\gamma$$

12. $$I=\lim_{\varepsilon\downarrow 0}\int_1^\infty x^{-\sigma}\log^m x\,\mathrm{e}^{-\varepsilon x^\alpha+\mathrm{i}(x^\alpha+\lambda x^\beta)}\,\mathrm{d}x,\qquad m\in\mathbb{N}_0,\quad \sigma,\lambda\in\mathbb{R},\quad \alpha,\beta>0,\quad \alpha/\beta\notin\mathbb{Q}$$

13. $$I_n(s;\mathbf{a})=\lim_{\varepsilon\downarrow 0}\int_0^\infty x^{s-1}\mathrm{e}^{-\varepsilon x^n+\mathrm{i}(x^n+a_{n-1}x^{n-1}+\cdots+a_1x)}\,\mathrm{d}x,\qquad n\geq 2,\quad \mathbf{a}\in\mathbb{R}^{n-1},\quad \Re s>0$$

14. $$I=\int_0^1\frac{\log\Gamma(x)\log\Gamma(1-x)}{x(1-x)}\,\mathrm{d}x$$

15. $$I_{mnr}=\mathrm{F.p.}\int_{-\infty}^\infty\frac{\psi^{(m)}(a+\mathrm{i}x)\psi^{(n)}(b-\mathrm{i}x)}{(x-c)^r}\,\mathrm{d}x,\qquad (m,n,r)\in\mathbb{N}_0^2\times\mathbb{N},\quad a,b>0,\quad c\in\mathbb{R}$$

16. $$I_m(a,b;f)=\int_0^\infty\frac{f(ax)-f(bx)}{x}\log^m x\,\mathrm{d}x,\qquad a,b>0,\quad m\in\mathbb{N}_0,\quad f\in\mathcal{S}(\mathbb{R}_+)$$

17. $$\int x\,\mathrm{d}x\qquad\text{(intermission)}$$

18. $$I=\int_0^1\lim_{N\to\infty}\sqrt{x+\sqrt{x^2+\sqrt{\cdots+\sqrt{x^N}}}}\,\mathrm{d}x$$

19. $$I_N(t;A,B)=\int_{U(N)}\mathrm{e}^{t\,\mathrm{Tr}(AUBU^\dagger)}\,\mathrm{d}U,$$
$$A=\mathrm{diag}(a_1,\ldots,a_N),\quad B=\mathrm{diag}(b_1,\ldots,b_N),$$
$$\mathbf{a},\mathbf{b}\in\mathbb{R}^N,\quad t\in\mathbb{C},\quad \mathrm{d}U(U(N))=1$$

20. $$I_N(g)=\frac{1}{N!}\int_{\mathbb{R}^N}\mathrm{e}^{-\sum_{j=1}^N(x_j^2/2+gx_j^4)}\prod_{1\leq i<j\leq N}(x_i-x_j)^2\,\mathrm{d}^N\mathbf{x},\qquad N\in\mathbb{N},\quad g>0$$

21. $$I_N(\lambda)=\int_{U(N)}\exp\left[\frac{\lambda}{2}\mathrm{Tr}(U+U^\dagger)\right]\mathrm{d}U,\qquad N\in\mathbb{N},\quad \lambda\in\mathbb{R},\quad \mathrm{d}U(U(N))=1$$

22. $$I_N(g)=\frac{1}{N!}\int_{\mathbb{R}^N}\prod_{j=1}^N\left(\frac{\mathrm{d}x_j}{2\pi}\mathrm{e}^{-x_j^2/(2g)}\right)\prod_{1\leq i<j\leq N}\left(2\sinh\frac{x_i-x_j}{2}\right)^2,\qquad N\in\mathbb{N},\quad g>0$$

23. $$I=\int a^n\,\mathrm{d}a\qquad\text{(this is insane)}$$

24. $$I_n=\int_{[0,1]^3}\frac{[x(1-x)y(1-y)z(1-z)]^n}{[1-(1-xy)z]^{n+1}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,\qquad n\in\mathbb{N}_0$$

25. $$I'_{abc} = \int_0^\infty t^c I_0(t)^a K_0(t)^b \, \mathrm{d}t, \qquad a, b \in \mathbb{N}_0, \quad b > a, \quad \Re c > -1$$

26. $$I(z) = \int_{[0,1]^4} \frac{\mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 \, \mathrm{d}x_4}{1 - 256z \prod_{j=1}^4 x_j(1-x_j)}, \qquad 0 \le z \le 1$$

27. $$I_p(k; f) = \int_{\mathbb{Z}_p^n} \exp\bigl(2\pi \mathrm{i} \{p^{-k} f(\mathbf{x})\}_p\bigr) \, \mathrm{d}\mu(\mathbf{x}),$$
$$p \text{ prime}, \quad k \in \mathbb{N}, \quad f \in \mathbb{Z}_p[x_1, \ldots, x_n], \quad \mu(\mathbb{Z}_p^n) = 1$$

28. $$I_{pq} = \int_0^1 \log^p G(1+x) \, \log^q G(2-x) \, \mathrm{d}x, \qquad (p, q) \in \mathbb{N}_0^2$$

29. $$I = \int_{[0,1]^3} \frac{\log^2(1-xyz)}{(1-xy)(1-yz)} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

30. $$I = \int_{[0,1]^4} \frac{\log(1-x_1x_2)\log(1-x_3x_4)}{1-x_1x_2x_3x_4} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 \, \mathrm{d}x_4$$

31. $$I = \int_0^\infty \frac{\log^4 x}{(1+x^2)(1+x^4)} \, \mathrm{d}x$$

32. $$I = \int_0^{\pi/3} \arccos\left(\frac{1-\cos x}{2\cos x}\right) \mathrm{d}x$$

33. $$I = \int_0^1 \frac{\log(1+x)\log(1-x)}{x(1+x^2)} \, \mathrm{d}x$$

34. $$I(\alpha, \beta, a, b, n, m) = \int_0^1 x^\alpha (1-x)^\beta \log^n(ax) \log^m(b(1-x)) \, \mathrm{d}x,$$
$$\alpha, \beta > -1, \quad a, b > 0, \quad (n, m) \in \mathbb{N}_0^2$$

35. $$I = \int_0^1 \frac{\log^2 x \, \log^2(1-x)}{1+x+x^2} \, \mathrm{d}x$$

36. $$I = \int_{\substack{x \ge 0, \; y \ge 0 \\ x+y \le 1}} \frac{\mathrm{d}x \, \mathrm{d}y}{xy + y(1-x-y) + x(1-x-y) + xy(1-x-y)}$$

37. $$I_3 = \frac{1}{\pi^3} \int_{[0,\pi]^3} \frac{\mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z}{1 - \frac{1}{3}(\cos x + \cos y + \cos z)}$$

38. $$I = \int_{\substack{x,y,z\geq 0\\ x+y+z\leq 1}} \frac{\mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z}{\sqrt{xyz(1-x-y-z)\,[1-256xyz(1-x-y-z)]}}$$

39. $$I_\lambda(\alpha,\beta,\gamma) = \int_{[0,1]^n} P_\lambda^{(1/\gamma)}(\mathbf{x}) \prod_{i=1}^{n} x_i^{\alpha-1}(1-x_i)^{\beta-1} \prod_{1\leq j<k\leq n} |x_j - x_k|^{2\gamma}\ \mathrm{d}^n\mathbf{x},$$
$$\alpha,\beta,\gamma > 0, \qquad \lambda \vdash m, \quad \ell(\lambda) \leq n$$

40. $$I = \int_0^\infty \frac{\log(1+\mathrm{e}^{-2\pi x})}{1+x^2}\ \mathrm{d}x$$

41. $$I = \int_{[0,2\pi]^n} \prod_{j<k} |\mathrm{e}^{\mathrm{i}\theta_j} - \mathrm{e}^{\mathrm{i}\theta_k}|^{2\beta} \prod_{j=1}^{n}\prod_{r=1}^{R} |\mathrm{e}^{\mathrm{i}\theta_j} - \mathrm{e}^{\mathrm{i}\phi_r}|^{2\alpha_r} \mathrm{e}^{\mathrm{i}\kappa_r \operatorname{Arg}(\mathrm{e}^{\mathrm{i}\theta_j} - \mathrm{e}^{\mathrm{i}\phi_r})}\ \mathrm{d}^n\boldsymbol{\theta},$$
$$\beta > 0, \quad \kappa_r \in \mathbb{R}, \quad \Re\alpha_r > -\frac{1}{2}, \quad \phi_r \not\equiv \phi_s \pmod{2\pi} \quad (r \neq s)$$

42. $$\{G(1), \mathcal{L}_{\min}\}, \qquad G(\lambda) = \int_0^1\int_0^1 \frac{\mathrm{d}x\ \mathrm{d}y}{\sqrt{x(1-x)y(1-y)(1-\lambda xy)}},$$
$$0 \leq \lambda \leq 1, \quad \mathcal{L}_{\min} \in \mathbb{Q}(\lambda)\langle\partial_\lambda\rangle,$$
$$\mathcal{L}_{\min}G = 0, \quad \mathcal{L}_{\min} \text{ monic, minimal}$$

43. $$I = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \log|1 + 2\cos\theta + 2\cos\phi|\ \mathrm{d}\theta\ \mathrm{d}\phi$$

44. $$I_D(Q^2) = \int_{\mathbb{R}^{2D}} \frac{\mathrm{d}^D k_1\ \mathrm{d}^D k_2}{\pi^D(k_1^2+m_1^2)(k_2^2+m_2^2)((p-k_1-k_2)^2+m_3^2)},$$
$$p^2 = Q^2 > 0, \quad (m_1,m_2,m_3) \in \mathbb{R}^3_{>0}, \quad 0 < \Re D < 3$$

45. (under.... construction)

46. $$I = \int_0^\infty\int_0^\infty \frac{\mathrm{d}x\ \mathrm{d}y}{xy + (1+x+y)(x+y+xy)m^2}, \qquad m > 0$$

47. $$I = \int_{-2}^{2}\int_{-2}^{2} \log^2|x-y|\ \frac{\sqrt{4-x^2}}{2\pi}\frac{\sqrt{4-y^2}}{2\pi}\ \mathrm{d}x\ \mathrm{d}y$$

48. $$I = \int_0^\infty \frac{\arctan(x)\log(1+x+x^2)}{x(1+x^2)}\ \mathrm{d}x$$

49. $$I = \int_0^{\pi/2} \frac{\log\left(1+\sqrt{\sin x\ \cos x}\right)}{\sqrt{\sin x}+\sqrt{\cos x}}\ \mathrm{d}x$$

50. $$I_5\left(Q^2; m_1,\ldots,m_6;\varepsilon\right) = \mathrm{e}^{5\gamma_E\varepsilon}\mu^{10\varepsilon}\int_{\mathbb{R}^{5D}} \frac{\displaystyle\prod_{r=1}^{5}\frac{\mathrm{d}^D k_r}{\pi^{D/2}}}{\left[\displaystyle\prod_{r=1}^{5}(k_r^2+m_r^2)\right]\left[\left(p-\displaystyle\sum_{r=1}^{5}k_r\right)^2+m_6^2\right]},$$
$$D = 2-2\varepsilon, \quad |\varepsilon| \ll 1, \quad \mu > 0, \quad p^2 = Q^2 > 0, \quad (m_1,\ldots,m_6) \in \mathbb{R}^6_{>0}$$